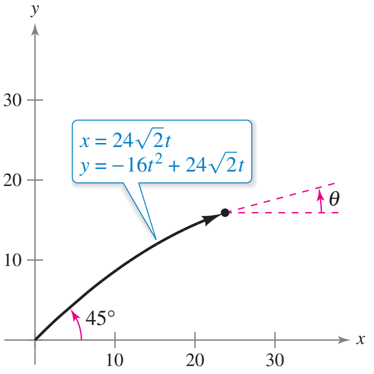
10.3 Parametric Equations and Calculus

- **Find the slope of a tangent line to a curve given by a set of parametric equations.**
- **Find the arc length of a curve given by a set of parametric equations.**
- **Find the area of a surface of revolution (parametric form).**

Slope and Tangent Lines

Now that you can represent a graph in the plane by a set of parametric equations, it is natural to ask how to use calculus to study plane curves. Consider the projectile represented by the parametric equations

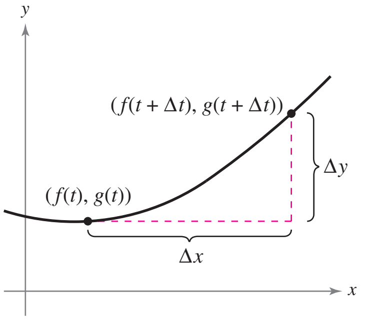
$$x = 24\sqrt{2}t \quad \text{and} \quad y = -16t^2 + 24\sqrt{2}t$$



At time t, the angle of elevation of the projectile is θ.
Figure 10.29

as shown in Figure 10.29. From the discussion at the beginning of Section 10.2, you know that these equations enable you to locate the position of the projectile at a given time. You also know that the object is initially projected at an angle of 45°, or a slope of $m = \tan 45° = 1$. But how can you find the slope at some other time t? The next theorem answers this question by giving a formula for the slope of the tangent line as a function of t.

> **THEOREM 10.7 Parametric Form of the Derivative**
>
> If a smooth curve C is given by the equations
>
> $$x = f(t) \quad \text{and} \quad y = g(t)$$
>
> then the slope of C at (x, y) is
>
> $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \quad \frac{dx}{dt} \neq 0.$$



The slope of the secant line through the points $(f(t), g(t))$ and $(f(t + \Delta t), g(t + \Delta t))$ is $\Delta y/\Delta x$.
Figure 10.30

Proof In Figure 10.30, consider $\Delta t > 0$ and let

$$\Delta y = g(t + \Delta t) - g(t) \quad \text{and} \quad \Delta x = f(t + \Delta t) - f(t).$$

Because $\Delta x \to 0$ as $\Delta t \to 0$, you can write

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta t \to 0} \frac{g(t + \Delta t) - g(t)}{f(t + \Delta t) - f(t)}.$$

Dividing both the numerator and denominator by Δt, you can use the differentiability of f and g to conclude that

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\Delta t \to 0} \frac{[g(t + \Delta t) - g(t)]/\Delta t}{[f(t + \Delta t) - f(t)]/\Delta t} \\ &= \frac{\displaystyle\lim_{\Delta t \to 0} \frac{g(t + \Delta t) - g(t)}{\Delta t}}{\displaystyle\lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}} \\ &= \frac{g'(t)}{f'(t)} \\ &= \frac{dy/dt}{dx/dt}.\end{aligned}$$

See LarsonCalculus.com for Bruce Edwards's video of this proof. ■

Exploration

The curve traced out in Example 1 is a circle. Use the formula

$$\frac{dy}{dx} = -\tan t$$

to find the slopes at the points (1, 0) and (0, 1).

EXAMPLE 1 Differentiation and Parametric Form

Find dy/dx for the curve given by $x = \sin t$ and $y = \cos t$.

Solution

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\sin t}{\cos t} = -\tan t$$

Because dy/dx is a function of t, you can use Theorem 10.7 repeatedly to find *higher-order* derivatives. For instance,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{d}{dx}\right] = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{dx/dt}$$ Second derivative

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left[\frac{d^2y}{dx^2}\right] = \frac{\frac{d}{dt}\left[\frac{d^2y}{dx^2}\right]}{dx/dt}.$$ Third derivative

EXAMPLE 2 Finding Slope and Concavity

For the curve given by

$$x = \sqrt{t} \quad \text{and} \quad y = \frac{1}{4}(t^2 - 4), \quad t \geq 0$$

find the slope and concavity at the point (2, 3).

Solution Because

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{(1/2)t}{(1/2)t^{-1/2}} = t^{3/2}$$ Parametric form of first derivative

you can find the second derivative to be

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}[dy/dx]}{dx/dt} = \frac{\frac{d}{dt}[t^{3/2}]}{dx/dt} = \frac{(3/2)t^{1/2}}{(1/2)t^{-1/2}} = 3t.$$ Parametric form of second derivative

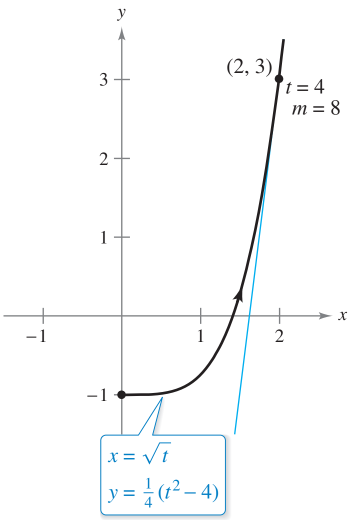
At $(x, y) = (2, 3)$, it follows that $t = 4$, and the slope is

$$\frac{dy}{dx} = (4)^{3/2} = 8.$$

Moreover, when $t = 4$, the second derivative is

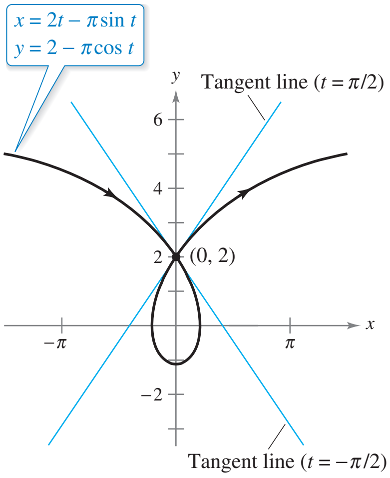
$$\frac{d^2y}{dx^2} = 3(4) = 12 > 0$$

and you can conclude that the graph is concave upward at (2, 3), as shown in Figure 10.31.



The graph is concave upward at (2, 3) when $t = 4$.
Figure 10.31

Because the parametric equations $x = f(t)$ and $y = g(t)$ need not define y as a function of x, it is possible for a plane curve to loop around and cross itself. At such points, the curve may have more than one tangent line, as shown in the next example.



This prolate cycloid has two tangent lines at the point $(0, 2)$.
Figure 10.32

EXAMPLE 3 **A Curve with Two Tangent Lines at a Point**

See LarsonCalculus.com for an interactive version of this type of example.

The **prolate cycloid** given by

$$x = 2t - \pi \sin t \quad \text{and} \quad y = 2 - \pi \cos t$$

crosses itself at the point $(0, 2)$, as shown in Figure 10.32. Find the equations of both tangent lines at this point.

Solution Because $x = 0$ and $y = 2$ when $t = \pm\pi/2$, and

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\pi \sin t}{2 - \pi \cos t}$$

you have $dy/dx = -\pi/2$ when $t = -\pi/2$ and $dy/dx = \pi/2$ when $t = \pi/2$. So, the two tangent lines at $(0, 2)$ are

$$y - 2 = -\left(\frac{\pi}{2}\right)x \qquad \text{Tangent line when } t = -\frac{\pi}{2}$$

and

$$y - 2 = \left(\frac{\pi}{2}\right)x. \qquad \text{Tangent line when } t = \frac{\pi}{2}$$

If $dy/dt = 0$ and $dx/dt \neq 0$ when $t = t_0$, then the curve represented by $x = f(t)$ and $y = g(t)$ has a horizontal tangent at $(f(t_0), g(t_0))$. For instance, in Example 3, the given curve has a horizontal tangent at the point $(0, 2 - \pi)$ (when $t = 0$). Similarly, if $dx/dt = 0$ and $dy/dt \neq 0$ when $t = t_0$, then the curve represented by $x = f(t)$ and $y = g(t)$ has a vertical tangent at $(f(t_0), g(t_0))$.

Arc Length

You have seen how parametric equations can be used to describe the path of a particle moving in the plane. You will now develop a formula for determining the *distance* traveled by the particle along its path.

Recall from Section 7.4 that the formula for the arc length of a curve C given by $y = h(x)$ over the interval $[x_0, x_1]$ is

$$\begin{aligned} s &= \int_{x_0}^{x_1} \sqrt{1 + [h'(x)]^2}\, dx \\ &= \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \end{aligned}$$

If C is represented by the parametric equations $x = f(t)$ and $y = g(t)$, $a \le t \le b$, and if $dx/dt = f'(t) > 0$, then

$$\begin{aligned} s &= \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\, dx \\ &= \int_a^b \sqrt{\frac{(dx/dt)^2 + (dy/dt)^2}{(dx/dt)^2}}\, \frac{dx}{dt}\, dt \\ &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt. \end{aligned}$$

THEOREM 10.8 Arc Length in Parametric Form

If a smooth curve C is given by $x = f(t)$ and $y = g(t)$ such that C does not intersect itself on the interval $a \le t \le b$ (except possibly at the endpoints), then the arc length of C over the interval is given by

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

REMARK When applying the arc length formula to a curve, be sure that the curve is traced out only once on the interval of integration. For instance, the circle given by $x = \cos t$ and $y = \sin t$ is traced out once on the interval $0 \le t \le 2\pi$, but is traced out twice on the interval $0 \le t \le 4\pi$.

In the preceding section, you saw that if a circle rolls along a line, then a point on its circumference will trace a path called a cycloid. If the circle rolls around the circumference of another circle, then the path of the point is an **epicycloid.** The next example shows how to find the arc length of an epicycloid.

ARCH OF A CYCLOID

The arc length of an arch of a cycloid was first calculated in 1658 by British architect and mathematician Christopher Wren, famous for rebuilding many buildings and churches in London, including St. Paul's Cathedral.

EXAMPLE 4 Finding Arc Length

A circle of radius 1 rolls around the circumference of a larger circle of radius 4, as shown in Figure 10.33. The epicycloid traced by a point on the circumference of the smaller circle is given by

$$x = 5\cos t - \cos 5t \quad \text{and} \quad y = 5\sin t - \sin 5t.$$

Find the distance traveled by the point in one complete trip about the larger circle.

Solution Before applying Theorem 10.8, note in Figure 10.33 that the curve has sharp points when $t = 0$ and $t = \pi/2$. Between these two points, dx/dt and dy/dt are not simultaneously 0. So, the portion of the curve generated from $t = 0$ to $t = \pi/2$ is smooth. To find the total distance traveled by the point, you can find the arc length of that portion lying in the first quadrant and multiply by 4.

$$s = 4\int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad \text{Parametric form for arc length}$$

$$= 4\int_0^{\pi/2} \sqrt{(-5\sin t + 5\sin 5t)^2 + (5\cos t - 5\cos 5t)^2}\, dt$$

$$= 20\int_0^{\pi/2} \sqrt{2 - 2\sin t \sin 5t - 2\cos t \cos 5t}\, dt$$

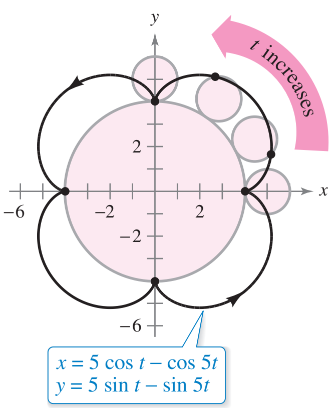
$$= 20\int_0^{\pi/2} \sqrt{2 - 2\cos 4t}\, dt \qquad \text{Difference formula for cosine}$$

$$= 20\int_0^{\pi/2} \sqrt{4\sin^2 2t}\, dt \qquad \text{Double-angle formula}$$

$$= 40\int_0^{\pi/2} \sin 2t\, dt$$

$$= -20\Big[\cos 2t\Big]_0^{\pi/2}$$

$$= 40$$



An epicycloid is traced by a point on the smaller circle as it rolls around the larger circle.
Figure 10.33

For the epicycloid shown in Figure 10.33, an arc length of 40 seems about right because the circumference of a circle of radius 6 is

$$2\pi r = 12\pi \approx 37.7.$$

Area of a Surface of Revolution

You can use the formula for the area of a surface of revolution in rectangular form to develop a formula for surface area in parametric form.

THEOREM 10.9 Area of a Surface of Revolution

If a smooth curve C given by $x = f(t)$ and $y = g(t)$ does not cross itself on an interval $a \le t \le b$, then the area S of the surface of revolution formed by revolving C about the coordinate axes is given by the following.

1. $S = 2\pi \int_a^b g(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ Revolution about the x-axis: $g(t) \ge 0$

2. $S = 2\pi \int_a^b f(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ Revolution about the y-axis: $f(t) \ge 0$

These formulas may be easier to remember if you think of the differential of arc length as

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Then the formulas are written as follows.

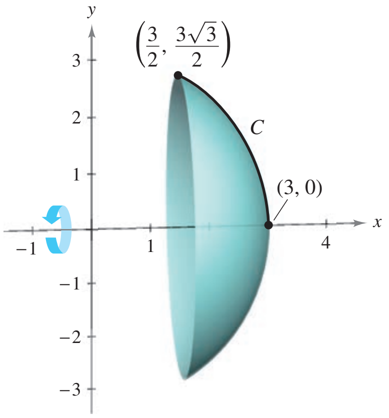
1. $S = 2\pi \int_a^b g(t)\, ds$ **2.** $S = 2\pi \int_a^b f(t)\, ds$

EXAMPLE 5 Finding the Area of a Surface of Revolution

Let C be the arc of the circle $x^2 + y^2 = 9$ from $(3, 0)$ to

$$\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$$

as shown in Figure 10.34. Find the area of the surface formed by revolving C about the x-axis.



The surface of revolution has a surface area of 9π.

Figure 10.34

Solution You can represent C parametrically by the equations

$$x = 3\cos t \quad \text{and} \quad y = 3\sin t, \quad 0 \le t \le \pi/3.$$

(Note that you can determine the interval for t by observing that $t = 0$ when $x = 3$ and $t = \pi/3$ when $x = 3/2$.) On this interval, C is smooth and y is nonnegative, and you can apply Theorem 10.9 to obtain a surface area of

$$S = 2\pi \int_0^{\pi/3} (3\sin t)\sqrt{(-3\sin t)^2 + (3\cos t)^2}\, dt$$ Formula for area of a surface of revolution

$$= 6\pi \int_0^{\pi/3} \sin t\sqrt{9(\sin^2 t + \cos^2 t)}\, dt$$

$$= 6\pi \int_0^{\pi/3} 3\sin t\, dt$$ Trigonometric identity

$$= -18\pi\Big[\cos t\Big]_0^{\pi/3}$$

$$= -18\pi\left(\frac{1}{2} - 1\right)$$

$$= 9\pi.$$

10.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Finding a Derivative In Exercises 1–4, find dy/dx.

1. $x = t^2, \; y = 7 - 6t$
2. $x = \sqrt[3]{t}, \; y = 4 - t$
3. $x = \sin^2 \theta, \; y = \cos^2 \theta$
4. $x = 2e^{\theta}, \; y = e^{-\theta/2}$

Finding Slope and Concavity In Exercises 5–14, find dy/dx and d^2y/dx^2, and find the slope and concavity (if possible) at the given value of the parameter.

Parametric Equations	Parameter
5. $x = 4t, \; y = 3t - 2$	$t = 3$
6. $x = \sqrt{t}, \; y = 3t - 1$	$t = 1$
7. $x = t + 1, \; y = t^2 + 3t$	$t = -1$
8. $x = t^2 + 5t + 4, \; y = 4t$	$t = 0$
9. $x = 4 \cos \theta, \; y = 4 \sin \theta$	$\theta = \frac{\pi}{4}$
10. $x = \cos \theta, \; y = 3 \sin \theta$	$\theta = 0$
11. $x = 2 + \sec \theta, \; y = 1 + 2 \tan \theta$	$\theta = \frac{\pi}{6}$
12. $x = \sqrt{t}, \; y = \sqrt{t - 1}$	$t = 2$
13. $x = \cos^3 \theta, \; y = \sin^3 \theta$	$\theta = \frac{\pi}{4}$
14. $x = \theta - \sin \theta, \; y = 1 - \cos \theta$	$\theta = \pi$

Finding Equations of Tangent Lines In Exercises 15–18, find an equation of the tangent line at each given point on the curve.

15. $x = 2 \cot \theta, \quad y = 2 \sin^2 \theta,$
$\left(-\frac{2}{\sqrt{3}}, \frac{3}{2}\right), (0, 2), \left(2\sqrt{3}, \frac{1}{2}\right)$
16. $x = 2 - 3 \cos \theta, \quad y = 3 + 2 \sin \theta,$
$(-1, 3), (2, 5), \left(\frac{4 + 3\sqrt{3}}{2}, 2\right)$
17. $x = t^2 - 4, \quad y = t^2 - 2t, \quad (0, 0), (-3, -1), (-3, 3)$
18. $x = t^4 + 2, \quad y = t^3 + t, \quad (2, 0), (3, -2), (18, 10)$

Finding an Equation of a Tangent Line In Exercises 19–22, (a) use a graphing utility to graph the curve represented by the parametric equations, (b) use a graphing utility to find dx/dt, dy/dt, and dy/dx at the given value of the parameter, (c) find an equation of the tangent line to the curve at the given value of the parameter, and (d) use a graphing utility to graph the curve and the tangent line from part (c).

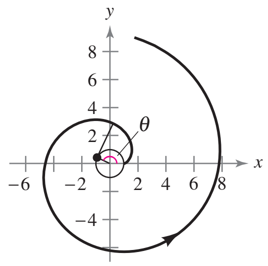
Parametric Equations	Parameter
19. $x = 6t, \; y = t^2 + 4$	$t = 1$
20. $x = t - 2, \; y = \frac{1}{t} + 3$	$t = 1$
21. $x = t^2 - t + 2, \; y = t^3 - 3t$	$t = -1$
22. $x = 3t - t^2, \quad y = 2t^{3/2}$	$t = \frac{1}{4}$

Finding Equations of Tangent Lines In Exercises 23–26, find the equations of the tangent lines at the point where the curve crosses itself.

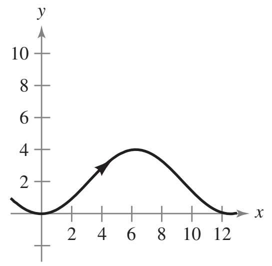
23. $x = 2 \sin 2t, \quad y = 3 \sin t$
24. $x = 2 - \pi \cos t, \quad y = 2t - \pi \sin t$
25. $x = t^2 - t, \quad y = t^3 - 3t - 1$
26. $x = t^3 - 6t, \quad y = t^2$

Horizontal and Vertical Tangency In Exercises 27 and 28, find all points (if any) of horizontal and vertical tangency to the portion of the curve shown.

27. Involute of a circle:
$x = \cos \theta + \theta \sin \theta$
$y = \sin \theta - \theta \cos \theta$



28. $x = 2\theta$
$y = 2(1 - \cos \theta)$



Horizontal and Vertical Tangency In Exercises 29–38, find all points (if any) of horizontal and vertical tangency to the curve. Use a graphing utility to confirm your results.

29. $x = 4 - t, \quad y = t^2$
30. $x = t + 1, \quad y = t^2 + 3t$
31. $x = t + 4, \quad y = t^3 - 3t$
32. $x = t^2 - t + 2, \quad y = t^3 - 3t$
33. $x = 3 \cos \theta, \quad y = 3 \sin \theta$
34. $x = \cos \theta, \quad y = 2 \sin 2\theta$
35. $x = 5 + 3 \cos \theta, \quad y = -2 + \sin \theta$
36. $x = 4 \cos^2 \theta, \quad y = 2 \sin \theta$
37. $x = \sec \theta, \quad y = \tan \theta$
38. $x = \cos^2 \theta, \quad y = \cos \theta$

Determining Concavity In Exercises 39–44, determine the open t-intervals on which the curve is concave downward or concave upward.

39. $x = 3t^2, \quad y = t^3 - t$
40. $x = 2 + t^2, \quad y = t^2 + t^3$
41. $x = 2t + \ln t, \quad y = 2t - \ln t$
42. $x = t^2, \quad y = \ln t$
43. $x = \sin t, \quad y = \cos t, \quad 0 < t < \pi$
44. $x = 4 \cos t, \quad y = 2 \sin t, \quad 0 < t < 2\pi$

Arc Length **In Exercises 45–50, find the arc length of the curve on the given interval.**

Parametric Equations	Interval
45. $x = 3t + 5, \quad y = 7 - 2t$	$-1 \le t \le 3$
46. $x = 6t^2, \quad y = 2t^3$	$1 \le t \le 4$
47. $x = e^{-t}\cos t, \quad y = e^{-t}\sin t$	$0 \le t \le \frac{\pi}{2}$
48. $x = \arcsin t, \quad y = \ln\sqrt{1 - t^2}$	$0 \le t \le \frac{1}{2}$
49. $x = \sqrt{t}, \quad y = 3t - 1$	$0 \le t \le 1$
50. $x = t, \quad y = \frac{t^5}{10} + \frac{1}{6t^3}$	$1 \le t \le 2$

Arc Length **In Exercises 51–54, find the arc length of the curve on the interval $[0, 2\pi]$.**

51. Hypocycloid perimeter: $x = a\cos^3\theta, y = a\sin^3\theta$

52. Circle circumference: $x = a\cos\theta, y = a\sin\theta$

53. Cycloid arch: $x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$

54. Involute of a circle: $x = \cos\theta + \theta\sin\theta, y = \sin\theta - \theta\cos\theta$

55. Path of a Projectile The path of a projectile is modeled by the parametric equations

$$x = (90\cos 30°)t \quad \text{and} \quad y = (90\sin 30°)t - 16t^2$$

where x and y are measured in feet.

(a) Use a graphing utility to graph the path of the projectile.

(b) Use a graphing utility to approximate the range of the projectile.

(c) Use the integration capabilities of a graphing utility to approximate the arc length of the path. Compare this result with the range of the projectile.

56. Path of a Projectile When the projectile in Exercise 55 is launched at an angle θ with the horizontal, its parametric equations are

$$x = (90\cos\theta)t \quad \text{and} \quad y = (90\sin\theta)t - 16t^2.$$

Use a graphing utility to find the angle that maximizes the range of the projectile. What angle maximizes the arc length of the trajectory?

57. Folium of Descartes Consider the parametric equations

$$x = \frac{4t}{1 + t^3} \quad \text{and} \quad y = \frac{4t^2}{1 + t^3}.$$

(a) Use a graphing utility to graph the curve represented by the parametric equations.

(b) Use a graphing utility to find the points of horizontal tangency to the curve.

(c) Use the integration capabilities of a graphing utility to approximate the arc length of the closed loop. (*Hint:* Use symmetry and integrate over the interval $0 \le t \le 1$.)

58. Witch of Agnesi Consider the parametric equations

$$x = 4\cot\theta \quad \text{and} \quad y = 4\sin^2\theta, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.$$

(a) Use a graphing utility to graph the curve represented by the parametric equations.

(b) Use a graphing utility to find the points of horizontal tangency to the curve.

(c) Use the integration capabilities of a graphing utility to approximate the arc length over the interval $\pi/4 \le \theta \le \pi/2$.

59. Writing

(a) Use a graphing utility to graph each set of parametric equations.

$$x = t - \sin t, \quad y = 1 - \cos t, \quad 0 \le t \le 2\pi$$

$$x = 2t - \sin(2t), \quad y = 1 - \cos(2t), \quad 0 \le t \le \pi$$

(b) Compare the graphs of the two sets of parametric equations in part (a). When the curve represents the motion of a particle and t is time, what can you infer about the average speeds of the particle on the paths represented by the two sets of parametric equations?

(c) Without graphing the curve, determine the time required for a particle to traverse the same path as in parts (a) and (b) when the path is modeled by

$$x = \tfrac{1}{2}t - \sin\left(\tfrac{1}{2}t\right) \quad \text{and} \quad y = 1 - \cos\left(\tfrac{1}{2}t\right).$$

60. Writing

(a) Each set of parametric equations represents the motion of a particle. Use a graphing utility to graph each set.

First Particle: $x = 3\cos t, \quad y = 4\sin t, \quad 0 \le t \le 2\pi$

Second Particle: $x = 4\sin t, \quad y = 3\cos t, \quad 0 \le t \le 2\pi$

(b) Determine the number of points of intersection.

(c) Will the particles ever be at the same place at the same time? If so, identify the point(s).

(d) Explain what happens when the motion of the second particle is represented by

$$x = 2 + 3\sin t, \quad y = 2 - 4\cos t, \quad 0 \le t \le 2\pi.$$

Surface Area **In Exercises 61–64, write an integral that represents the area of the surface generated by revolving the curve about the x-axis. Use a graphing utility to approximate the integral.**

Parametric Equations	Interval
61. $x = 3t, \quad y = t + 2$	$0 \le t \le 4$
62. $x = \frac{1}{4}t^2, \quad y = t + 3$	$0 \le t \le 3$
63. $x = \cos^2\theta, \quad y = \cos\theta$	$0 \le \theta \le \frac{\pi}{2}$
64. $x = \theta + \sin\theta, \quad y = \theta + \cos\theta$	$0 \le \theta \le \frac{\pi}{2}$

Surface Area **In Exercises 65–70, find the area of the surface generated by revolving the curve about each given axis.**

65. $x = 2t, \quad y = 3t, \quad 0 \le t \le 3$

(a) x-axis (b) y-axis

66. $x = t, \quad y = 4 - 2t, \quad 0 \le t \le 2$

(a) x-axis (b) y-axis

67. $x = 5 \cos \theta, \quad y = 5 \sin \theta, \quad 0 \le \theta \le \frac{\pi}{2}, \quad y$-axis

68. $x = \frac{1}{3}t^3, \quad y = t + 1, \quad 1 \le t \le 2, \quad y$-axis

69. $x = a \cos^3 \theta, \quad y = a \sin^3 \theta, \quad 0 \le \theta \le \pi, \quad x$-axis

70. $x = a \cos \theta, \quad y = b \sin \theta, \quad 0 \le \theta \le 2\pi$

(a) x-axis (b) y-axis

WRITING ABOUT CONCEPTS

71. Parametric Form of the Derivative Give the parametric form of the derivative.

Mental Math **In Exercises 72 and 73, mentally determine dy/dx.**

72. $x = t, \quad y = 3$

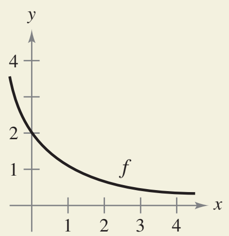
73. $x = t, \quad y = 6t - 5$

74. Arc Length Give the integral formula for arc length in parametric form.

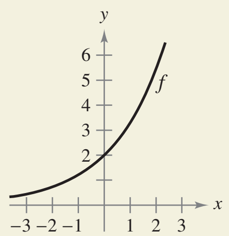
75. Surface Area Give the integral formulas for the areas of the surfaces of revolution formed when a smooth curve C is revolved about (a) the x-axis and (b) the y-axis.

76. HOW DO YOU SEE IT? Using the graph of f, (a) determine whether dy/dt is positive or negative given that dx/dt is negative, and (b) determine whether dx/dt is positive or negative given that dy/dt is positive. Explain your reasoning.

(i)



(ii)



77. Integration by Substitution Use integration by substitution to show that if y is a continuous function of x on the interval $a \le x \le b$, where $x = f(t)$ and $y = g(t)$, then

$$\int_a^b y\,dx = \int_{t_1}^{t_2} g(t) f'(t)\,dt$$

where $f(t_1) = a$, $f(t_2) = b$, and both g and f' are continuous on $[t_1, t_2]$.

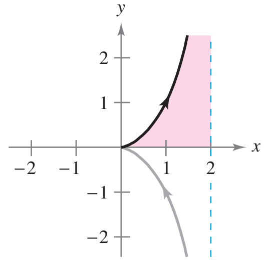
78. Surface Area A portion of a sphere of radius r is removed by cutting out a circular cone with its vertex at the center of the sphere. The vertex of the cone forms an angle of 2θ. Find the surface area removed from the sphere.

Area **In Exercises 79 and 80, find the area of the region. (Use the result of Exercise 77.)**

79. $x = 2 \sin^2 \theta$

$y = 2 \sin^2 \theta \tan \theta$

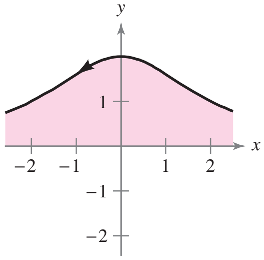
$0 \le \theta < \frac{\pi}{2}$



80. $x = 2 \cot \theta$

$y = 2 \sin^2 \theta$

$0 < \theta < \pi$



Areas of Simple Closed Curves **In Exercises 81–86, use a computer algebra system and the result of Exercise 77 to match the closed curve with its area. (These exercises were based on "The Surveyor's Area Formula" by Bart Braden, *College Mathematics Journal,* September 1986, pp. 335–337, by permission of the author.)**

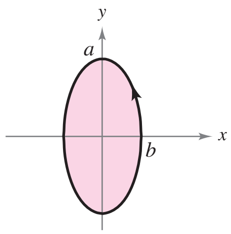
(a) $\frac{8}{3}ab$ **(b)** $\frac{3}{8}\pi a^2$ **(c)** $2\pi a^2$

(d) πab **(e)** $2\pi ab$ **(f)** $6\pi a^2$

81. Ellipse: $(0 \le t \le 2\pi)$

$x = b \cos t$

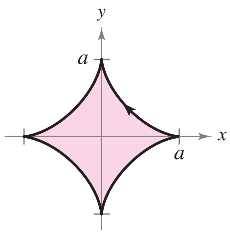
$y = a \sin t$



82. Astroid: $(0 \le t \le 2\pi)$

$x = a \cos^3 t$

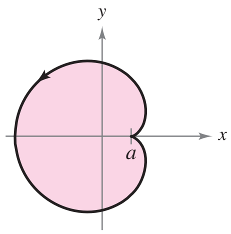
$y = a \sin^3 t$



83. Cardioid: $(0 \le t \le 2\pi)$

$x = 2a \cos t - a \cos 2t$

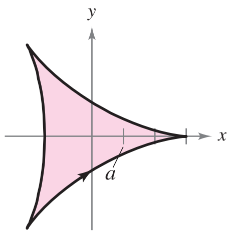
$y = 2a \sin t - a \sin 2t$



84. Deltoid: $(0 \le t \le 2\pi)$

$x = 2a \cos t + a \cos 2t$

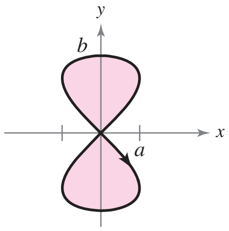
$y = 2a \sin t - a \sin 2t$



85. Hourglass: $(0 \le t \le 2\pi)$

$x = a \sin 2t$

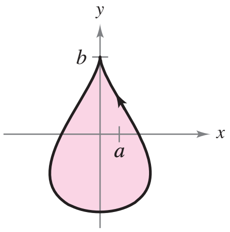
$y = b \sin t$



86. Teardrop: $(0 \le t \le 2\pi)$

$x = 2a \cos t - a \sin 2t$

$y = b \sin t$



Centroid **In Exercises 87 and 88, find the centroid of the region bounded by the graph of the parametric equations and the coordinate axes. (Use the result of Exercise 77.)**

87. $x = \sqrt{t}, \quad y = 4 - t$

88. $x = \sqrt{4 - t}, \quad y = \sqrt{t}$

Volume **In Exercises 89 and 90, find the volume of the solid formed by revolving the region bounded by the graphs of the given equations about the *x*-axis. (Use the result of Exercise 77.)**

89. $x = 6 \cos \theta, \quad y = 6 \sin \theta$

90. $x = \cos \theta, \quad y = 3 \sin \theta, \quad a > 0$

91. Cycloid Use the parametric equations

$$x = a(\theta - \sin \theta) \quad \text{and} \quad y = a(1 - \cos \theta),\ a > 0$$

to answer the following.

(a) Find dy/dx and d^2y/dx^2.

(b) Find the equation of the tangent line at the point where $\theta = \pi/6$.

(c) Find all points (if any) of horizontal tangency.

(d) Determine where the curve is concave upward or concave downward.

(e) Find the length of one arc of the curve.

92. Using Parametric Equations Use the parametric equations

$$x = t^2\sqrt{3} \quad \text{and} \quad y = 3t - \frac{1}{3}t^3$$

to answer the following.

(a) Use a graphing utility to graph the curve on the interval $-3 \le t \le 3$.

(b) Find dy/dx and d^2y/dx^2.

(c) Find the equation of the tangent line at the point $\left(\sqrt{3}, \frac{8}{3}\right)$.

(d) Find the length of the curve.

(e) Find the surface area generated by revolving the curve about the *x*-axis.

93. Involute of a Circle The involute of a circle is described by the endpoint P of a string that is held taut as it is unwound from a spool that does not turn (see figure). Show that a parametric representation of the involute is

$$x = r(\cos \theta + \theta \sin \theta) \quad \text{and} \quad y = r(\sin \theta - \theta \cos \theta).$$

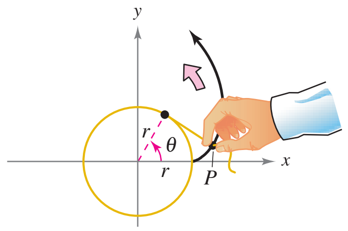


Figure for 93

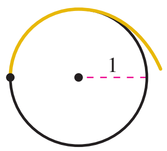


Figure for 94

94. Involute of a Circle The figure shows a piece of string tied to a circle with a radius of one unit. The string is just long enough to reach the opposite side of the circle. Find the area that is covered when the string is unwound counterclockwise.

95. Using Parametric Equations

(a) Use a graphing utility to graph the curve given by

$$x = \frac{1 - t^2}{1 + t^2} \quad \text{and} \quad y = \frac{2t}{1 + t^2}$$

where $-20 \le t \le 20$.

(b) Describe the graph and confirm your result analytically.

(c) Discuss the speed at which the curve is traced as t increases from -20 to 20.

96. Tractrix A person moves from the origin along the positive *y*-axis pulling a weight at the end of a 12-meter rope. Initially, the weight is located at the point $(12, 0)$.

(a) In Exercise 90 of Section 8.7, it was shown that the path of the weight is modeled by the rectangular equation

$$y = -12 \ln\left(\frac{12 - \sqrt{144 - x^2}}{x}\right) - \sqrt{144 - x^2}$$

where $0 < x \le 12$. Use a graphing utility to graph the rectangular equation.

(b) Use a graphing utility to graph the parametric equations

$$x = 12 \operatorname{sech} \frac{t}{12} \quad \text{and} \quad y = t - 12 \tanh \frac{t}{12}$$

where $t \ge 0$. How does this graph compare with the graph in part (a)? Which graph (if either) do you think is a better representation of the path?

(c) Use the parametric equations for the tractrix to verify that the distance from the *y*-intercept of the tangent line to the point of tangency is independent of the location of the point of tangency.

True or False? **In Exercises 97 and 98, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

97. If $x = f(t)$ and $y = g(t)$, then $\dfrac{d^2y}{dx^2} = \dfrac{g''(t)}{f''(t)}$.

98. The curve given by $x = t^3$, $y = t^2$ has a horizontal tangent at the origin because $dy/dt = 0$ when $t = 0$.